Tutorial 5 - ARIMA Models and Simulation

Understanding Model Components Through Simulation

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Learning Outcomes

> Understand the behavior of AR, MA, and ARMA processes through simulation Identify patterns in ACF and PACF that correspond to different models Apply the Box-Jenkins approach to model identification Begin the model selection process for financial time series

This tutorial is the second in a series of three that replace the previous HW04 assignment, providing a more structured approach aligned with the lecture material.

Quick Recap: Stationarity and Transformation

In Tutorial 5, we covered: - Stationarity: constant mean, variance, and autocorrelation structure - Identifying non-stationarity through time plots and ACF/PACF - Transformations to achieve stationarity including differencing and Box-Cox transformations

Now we'll build on these concepts to understand ARIMA model components and begin the model selection process.

Part 1: Understanding ARIMA Components Through Simulation

ARIMA (AutoRegressive Integrated Moving Average) models have three components: - AR: AutoRegressive terms (p) - I: Integrated/differencing terms (d) - MA: Moving Average terms (q)

Let's explore each component through simulation to understand their behavior.

Exercise 1: AutoRegressive (AR) Processes

AR models use past values of the series to predict future values. An AR(p) model is:

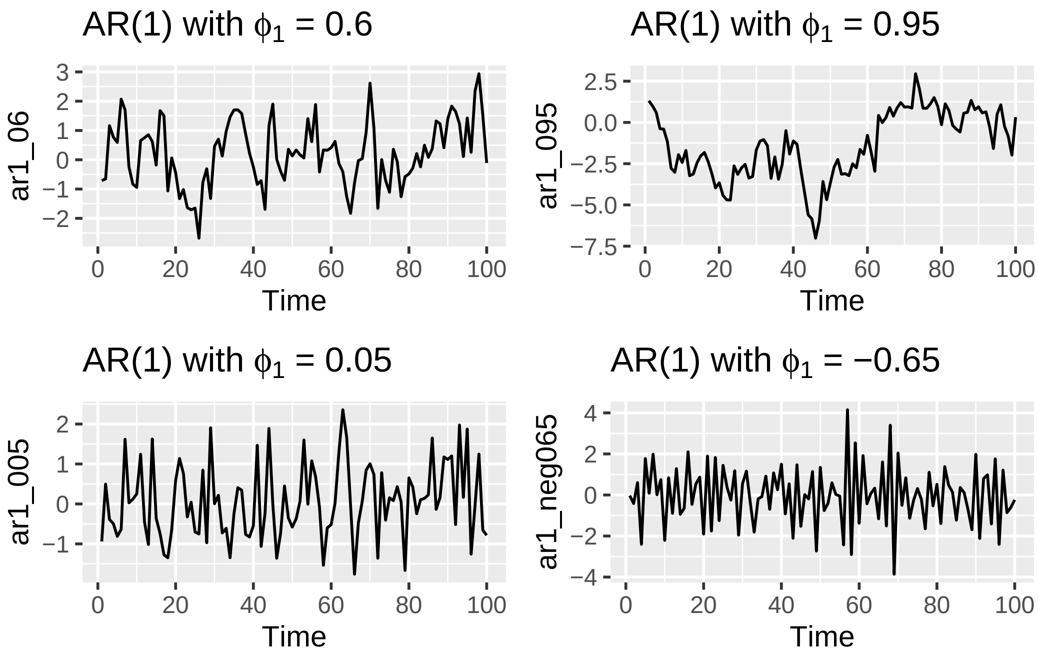
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t$$

First, let's create an AR(1) simulation function:

```
ar1 <- function(phi, n=100, mean=0) {
  y <- ts(numeric(n))
  e <- rnorm(n)
  y[1] <- rnorm(1)  # Random start
  for(i in 2:n)
    y[i] <- mean + phi*y[i-1] + e[i]
  return(y)
}
```

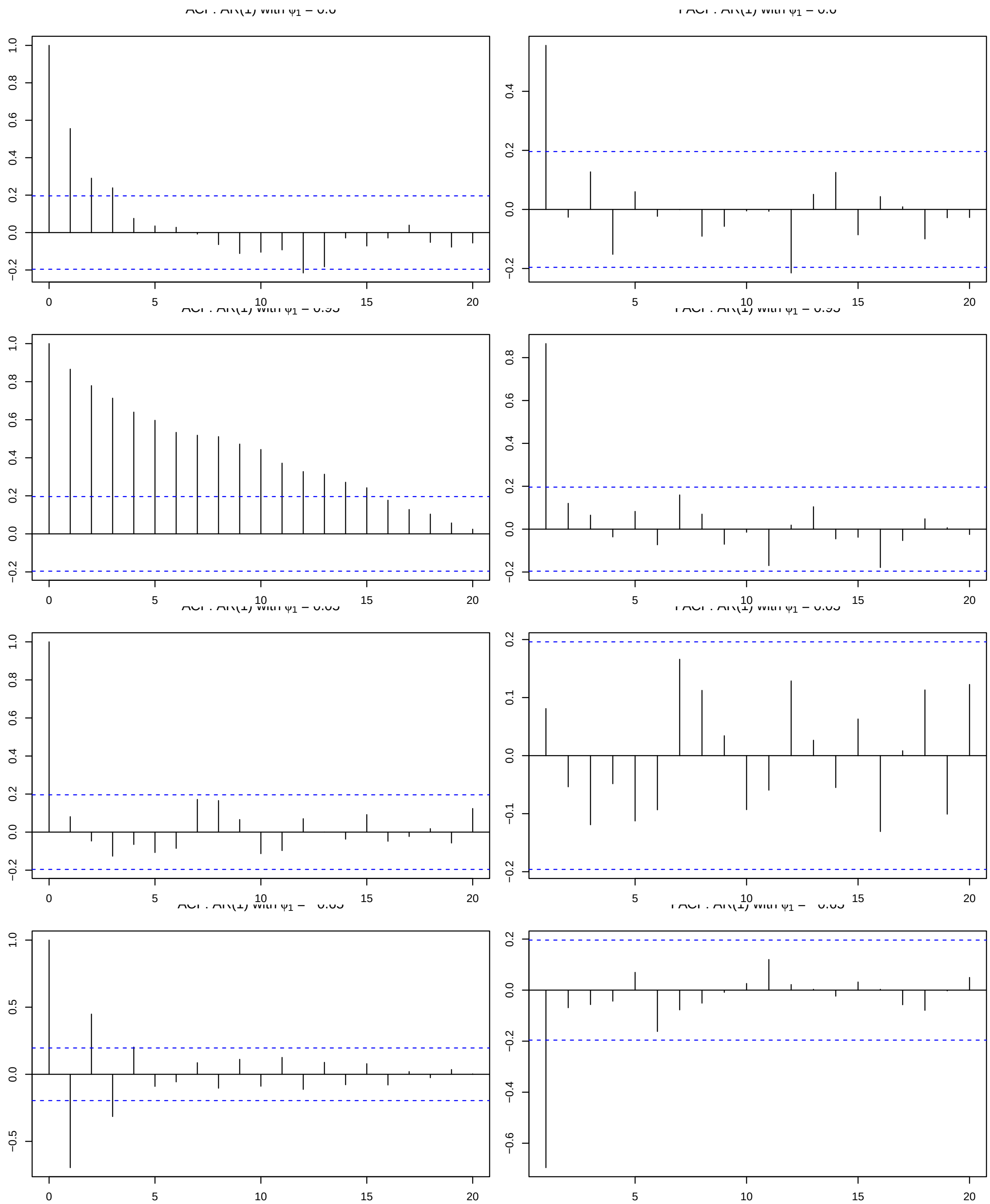
Now, let's simulate AR(1) models with different parameters:

```
# Create a 2x2 grid of simulations with different phi values
set.seed(123) # For reproducibility
ar1_06 <- ar1(0.6)
ar1_095 <- ar1(0.95)
ar1_005 <- ar1(0.05)
ar1_neg065 <- ar1(-0.65)

# AR(1) plots
p1 <- autoplot(ar1_06) + ggtitle(expression(paste("AR(1) with ", phi[1], " = 0.6")))
p2 <- autoplot(ar1_095) + ggtitle(expression(paste("AR(1) with ", phi[1], " = 0.95")))
p3 <- autoplot(ar1_005) + ggtitle(expression(paste("AR(1) with ", phi[1], " = 0.05")))
p4 <- autoplot(ar1_neg065) + ggtitle(expression(paste("AR(1) with ", phi[1], " = -0.65")))

grid.arrange(p1, p2, p3, p4, nrow=2)
```



Now let's examine the ACF and PACF patterns for these processes:

```
# Add this before your plotting code
par(mar=c(2,2,2,1))  # Reduce margins even more

# Then your existing plotting code
par(mfrow=c(4,2))
acf(ar1_06, main=expression(paste("ACF: AR(1) with ", phi[1], " = 0.6")))
pacf(ar1_06, main=expression(paste("PACF: AR(1) with ", phi[1], " = 0.6")))
acf(ar1_095, main=expression(paste("ACF: AR(1) with ", phi[1], " = 0.95")))
pacf(ar1_095, main=expression(paste("PACF: AR(1) with ", phi[1], " = 0.95")))
acf(ar1_005, main=expression(paste("ACF: AR(1) with ", phi[1], " = 0.05")))
pacf(ar1_005, main=expression(paste("PACF: AR(1) with ", phi[1], " = 0.05")))
acf(ar1_neg065, main=expression(paste("ACF: AR(1) with ", phi[1], " = -0.65")))
pacf(ar1_neg065, main=expression(paste("PACF: AR(1) with ", phi[1], " = -0.65")))
```



```r
par(mfrow=c(1,1))
```

a. Describe how the time series pattern changes as phi_1 varies.

Solution. **Solution:** The time series pattern changes significantly as the AR(1) coefficient phi_1 varies:

phi_1 = 0.6 (moderate positive autocorrelation): - Shows moderate persistence where values tend to stay on the same side of the mean for several periods - Displays relatively smooth transitions between values - Returns to the mean at a moderate rate

phi_1 = 0.95 (strong positive autocorrelation): - Exhibits very high persistence with long swings away from the mean - Shows the slowest mean reversion, creating long cycles - Resembles a random walk (though still technically stationary) - Creates the appearance of temporary trends

phi_1 = 0.05 (very weak autocorrelation): - Behaves almost like white noise with minimal persistence - Shows rapid fluctuations around the mean - Displays very little memory of previous values - Quick mean reversion

phi_1 = -0.65 (moderate negative autocorrelation): - Creates an oscillating pattern with frequent sign changes - Shows rapid alternation between positive and negative values - Tends to overcompensate for previous deviations from the mean - Creates a jagged, rapid-fluctuating appearance

The closer phi_1 is to 1, the more persistent the series becomes (approaching non-stationarity at phi_1 = 1). Negative values of phi_1 create oscillatory behavior, with larger negative values producing more pronounced oscillations.

b. For each value of phi_1, identify the pattern in the ACF and PACF.

Solution. **Solution:** The ACF and PACF patterns vary with different ϕ_1 values:

$\phi_1 = 0.6$**:** - ACF: Shows a gradual exponential decay ($\rho_k = 0.6^k$ for lag k) - PACF: Shows a sharp cutoff after lag 1 with a significant spike at lag 1 (0.6)

phi_1 = 0.95: - ACF: Shows a very slow exponential decay, remaining significant for many lags - PACF: Shows a sharp cutoff after lag 1 with a large spike at lag 1 (0.95)

phi_1 = 0.05: - ACF: Shows a very rapid decay, becoming insignificant after lag 1 - PACF: Shows a sharp cutoff after lag 1 with a small but significant spike at lag 1 (0.05)

phi_1 = -0.65: - ACF: Shows a damped oscillatory pattern, alternating between positive and negative values - PACF: Shows a sharp cutoff after lag 1 with a negative spike at lag 1 (-0.65)

These patterns confirm the theoretical properties of AR(1) models: the ACF decays exponentially at rate phi_1, while the PACF has a single significant spike at lag 1 equal to phi_1.

c. Based on the lecture, explain why the ACF "dies out" gradually while the PACF has a sharp cutoff at lag 1 for AR(1) processes.

Solution. **Solution:** The different patterns in ACF and PACF for AR(1) processes can be explained by their theoretical properties:

Why the ACF dies out gradually: In an AR(1) process defined as y_t = phi_1*y_t-1 + _t, the theoretical autocorrelation function is:

_k = phi_1^k for lag k 1

This is an exponential decay pattern. The correlation between observations k periods apart is the AR coefficient raised to the power k. When |phi_1| < 1 (required for stationarity), this value gradually decreases toward zero as k increases. The rate of decay depends on the magnitude of phi_1: - Values closer to 1 produce slower decay - Values closer to 0 produce faster decay - Negative values produce oscillating decay

Why the PACF has a sharp cutoff: The partial autocorrelation at lag k measures the correlation between y_t and y_t-k after removing the effects of the intermediate lags (y_t-1, y_t-2, …, y_t-k+1).

For an AR(1) process: - The correlation between y_t and y_t-1 is direct and equal to phi_1 - For lags k > 1, once we control for y_t-1, there is no direct connection between y_t and y_t-k

This happens because in an AR(1) process, y_t is only directly influenced by y_t-1, not by values from earlier periods. Any apparent correlation with earlier lags is fully explained by the chain of lag-1 relationships. Therefore, the PACF is phi_1 at lag 1 and 0 at all lags greater than 1.

This property is what makes PACF particularly useful for identifying the order (p) of AR processes.

d. Which of these models appear similar to patterns you might see in financial returns?

Solution. **Solution:** Among the simulated AR(1) models, the one with phi_1 = 0.05 most closely resembles typical financial return series for several reasons:

1. **Low persistence:** Financial returns typically show very little autocorrelation due to market efficiency, similar to the weak correlation in the phi_1 = 0.05 model.
2. **Mean reversion:** Returns tend to fluctuate around a central mean (often near zero), just as this model quickly reverts to its mean.
3. **Limited predictability:** The weak AR coefficient implies limited predictability from past values, consistent with the difficulty of forecasting financial returns.

4. **Rapid fluctuations:** The quick changes and limited memory in this model align with the rapid price adjustments in efficient markets.

This resemblance is consistent with the Efficient Market Hypothesis, which suggests returns should be largely unpredictable from past returns. However, it's worth noting that actual financial returns often exhibit:

- Heavier tails than normal distributions (extreme events are more common)
- Volatility clustering (periods of high volatility tend to persist)
- Occasional structural breaks

These features would require more complex models (like GARCH) to capture fully. The AR(1) with phi_1 = 0.05 captures the baseline weak autocorrelation structure but misses these more complex characteristics.

Exercise 2: Moving Average (MA) Processes

MA models use past errors to predict future values. An MA(q) model is:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q}$$

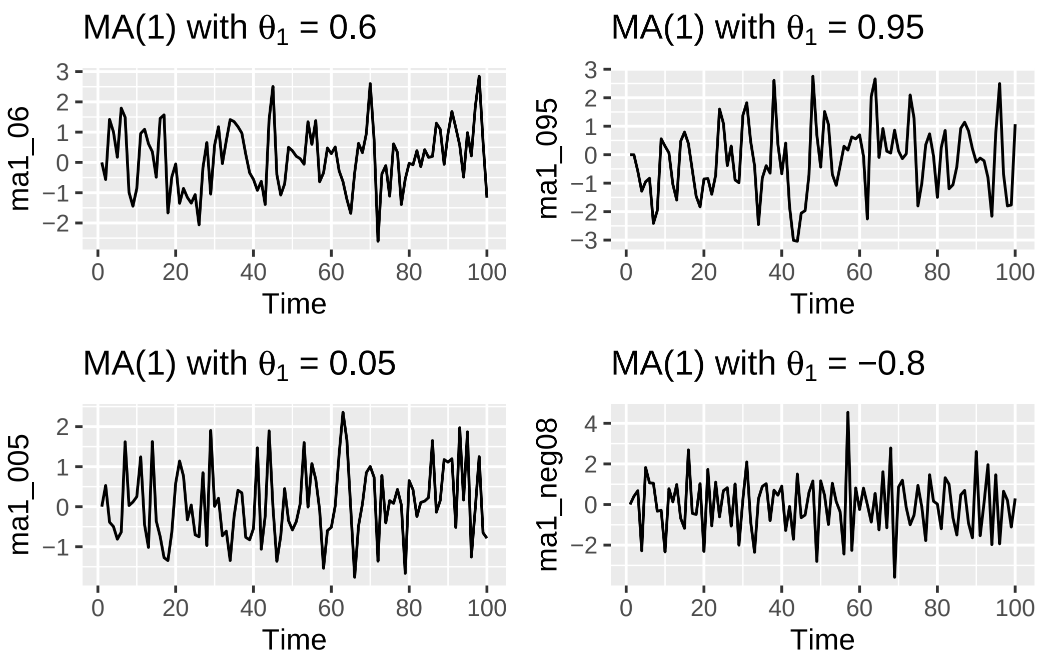
Let's create an MA(1) simulation function:

```
ma1 <- function(theta, n=100) {
  y <- ts(numeric(n))
  e <- rnorm(n+1) # Need n+1 because we use e[i-1]
  for(i in 2:n)
    y[i] <- theta*e[i-1] + e[i]
  return(y)
}
```

Now, let's simulate MA(1) models with different parameters:

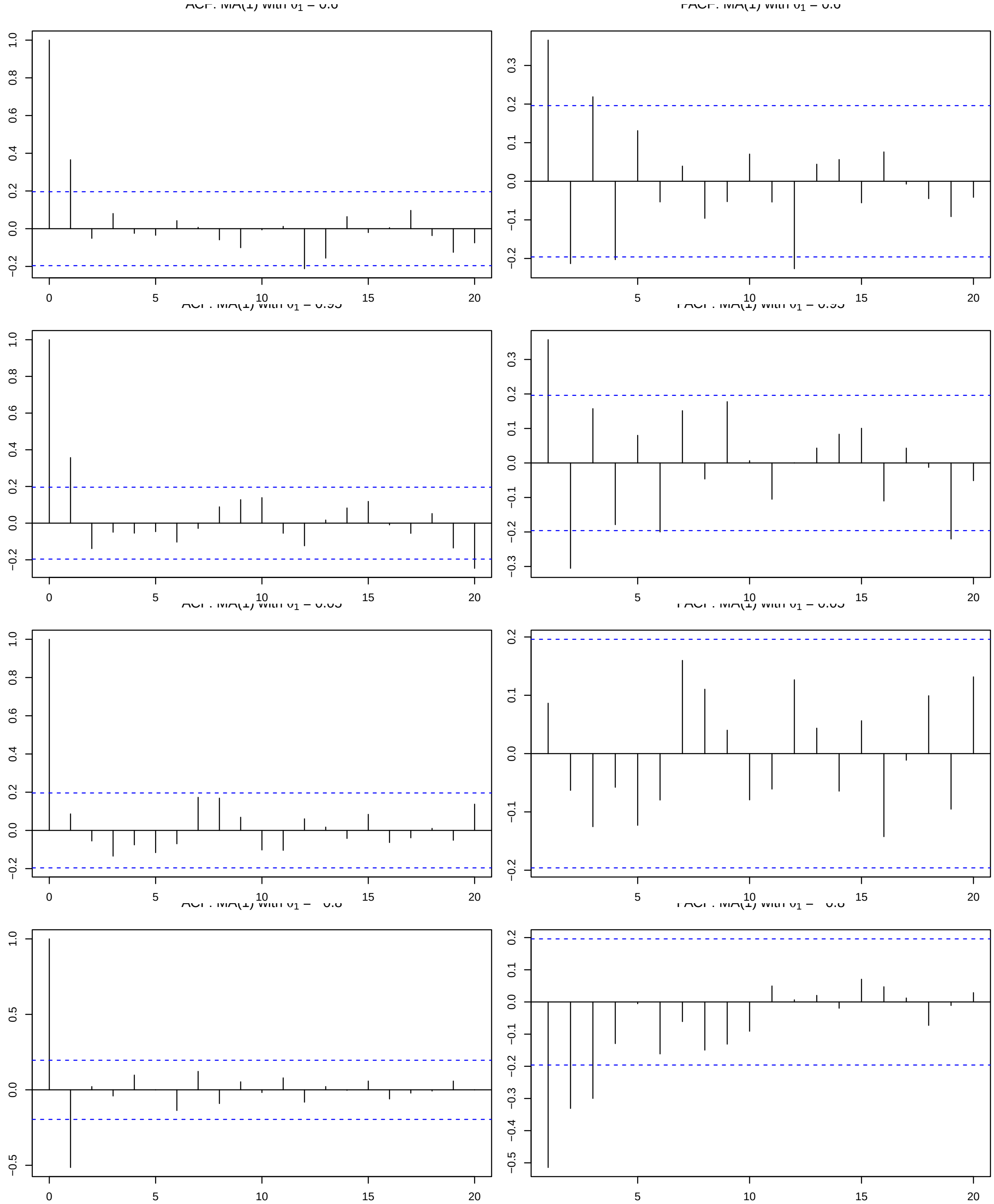
```
set.seed(123)
ma1_06 <- ma1(0.6)
ma1_095 <- ma1(0.95)
ma1_005 <- ma1(0.05)
ma1_neg08 <- ma1(-0.8)

# MA(1) plots
p1 <- autoplot(ma1_06) + ggtitle(expression(paste("MA(1) with ", theta[1], " = 0.6")))
p2 <- autoplot(ma1_095) + ggtitle(expression(paste("MA(1) with ", theta[1], " = 0.95")))
```

```r
p3 <- autoplot(ma1_005) + ggtitle(expression(paste("MA(1) with ", theta[1], " = 0.05")))
p4 <- autoplot(ma1_neg08) + ggtitle(expression(paste("MA(1) with ", theta[1], " = -0.8")))

grid.arrange(p1, p2, p3, p4, nrow=2)
```



And their ACF and PACF patterns:

```r
# Show ACF and PACF for each model
par(mar=c(2,2,2,1))  # Reduce margins even more: bottom, left, top, right
par(mfrow=c(4,2))
acf(ma1_06, main=expression(paste("ACF: MA(1) with ", theta[1], " = 0.6")))
pacf(ma1_06, main=expression(paste("PACF: MA(1) with ", theta[1], " = 0.6")))
acf(ma1_095, main=expression(paste("ACF: MA(1) with ", theta[1], " = 0.95")))
pacf(ma1_095, main=expression(paste("PACF: MA(1) with ", theta[1], " = 0.95")))
acf(ma1_005, main=expression(paste("ACF: MA(1) with ", theta[1], " = 0.05")))
pacf(ma1_005, main=expression(paste("PACF: MA(1) with ", theta[1], " = 0.05")))
acf(ma1_neg08, main=expression(paste("ACF: MA(1) with ", theta[1], " = -0.8")))
pacf(ma1_neg08, main=expression(paste("PACF: MA(1) with ", theta[1], " = -0.8")))
```



```r
par(mfrow=c(1,1))
```

a. How does the time series pattern change as theta_1 varies?

Solution. **Solution:** The time series pattern of MA(1) processes changes in the following ways as theta_1 varies:

theta_1 = 0.6 (moderate positive MA coefficient): - Shows limited memory where each observation is affected by the current and previous error - Displays more random-looking fluctuations than an AR process with the same coefficient - No persistent deviations from the mean

theta_1 = 0.95 (strong positive MA coefficient): - Creates stronger connection between consecutive observations - Still lacks the long persistent swings seen in AR models - Shows more pronounced short-term dependencies - Each error has a strong effect on two consecutive observations

theta_1 = 0.05 (very weak MA coefficient): - Behaves almost like pure white noise - Shows minimal impact of previous errors - Highly unpredictable pattern with almost no visible structure

theta_1 = -0.8 (strong negative MA coefficient): - Creates negative correlation between consecutive observations - Tends to show reversals (positive values followed by negative, and vice versa) - Produces a "jittery" pattern with frequent reversals

Unlike AR processes, MA processes don't show long-term persistence regardless of coefficient value, as the effect of each shock is limited to a finite number of future observations (in this case, just one future observation).

b. For each value of theta_1, identify the pattern in the ACF and PACF.

Solution. **Solution:** The ACF and PACF patterns vary distinctively with different theta_1 values:

theta_1 = 0.6: - ACF: Shows a single significant spike at lag 1 ($0.6/(1+0.6^2)$ 0.51) and cuts off to zero after lag 1 - PACF: Shows a gradually decaying pattern with the first value approximately 0.51, and subsequent values declining

theta_1 = 0.95: - ACF: Shows a single large significant spike at lag 1 (0.69) and cuts off to zero - PACF: Shows a slow decay with alternating signs, but remaining significant for many lags

theta_1 = 0.05: - ACF: Shows a very small spike at lag 1 (0.05) that may barely cross significance threshold - PACF: Shows very small values at all lags, resembling white noise

theta_1 = -0.8: - ACF: Shows a single significant negative spike at lag 1 (-0.62) and cuts off to zero - PACF: Shows a decaying pattern with alternating signs, starting with the negative value at lag 1

These patterns demonstrate the key characteristics of MA(1) processes: the ACF has a single significant spike at lag 1 and cuts off, while the PACF gradually decays. This is exactly the opposite pattern of AR(1) processes.

c. How does the ACF and PACF pattern of MA processes differ from AR processes?

Solution. **Solution:** The ACF and PACF patterns of MA and AR processes exhibit opposite behaviors:

MA Processes: - **ACF:** Shows a sharp cutoff after lag q (the order of the MA process) - **PACF:** Shows a gradual decay or damped oscillation

AR Processes: - **ACF:** Shows a gradual decay or damped oscillation - **PACF:** Shows a sharp cutoff after lag p (the order of the AR process)

This difference creates a clear identification pattern: 1. If the ACF cuts off sharply while the PACF decays gradually, this suggests an MA process 2. If the PACF cuts off sharply while the ACF decays gradually, this suggests an AR process

The theoretical explanation is: - In an MA(q) process, correlation is directly limited to q lags (the error term only affects q future observations) - In an AR(p) process, each observation directly affects only the next p observations, but indirectly affects all future observations through the recursive structure

This contrasting behavior is a fundamental tool for model identification in the Box-Jenkins methodology discussed in the lecture.

d. From these simulations, what's the key feature that helps you distinguish between AR and MA processes?

Solution. **Solution:** The key feature that helps distinguish between AR and MA processes is the contrasting behavior of their ACF and PACF functions:

Key Distinguishing Features:

1. **Pattern of "decay vs. cutoff":**
   - AR processes: ACF decays gradually while PACF cuts off after lag p
   - MA processes: ACF cuts off after lag q while PACF decays gradually
2. **Persistence in the time series:**
   - AR processes (especially with coefficients near 1) show persistent deviations from the mean

- MA processes show limited memory with no long-term persistence regardless of coefficient values

3. **Long-term behavior:**
   - AR processes can show long swings that appear trend-like in small samples
   - MA processes always revert quickly to the mean with no sustained movements

4. **Response to shocks:**
   - AR processes: shocks have diminishing but theoretically infinite effects
   - MA processes: shocks have larger immediate effects but disappear completely after q periods

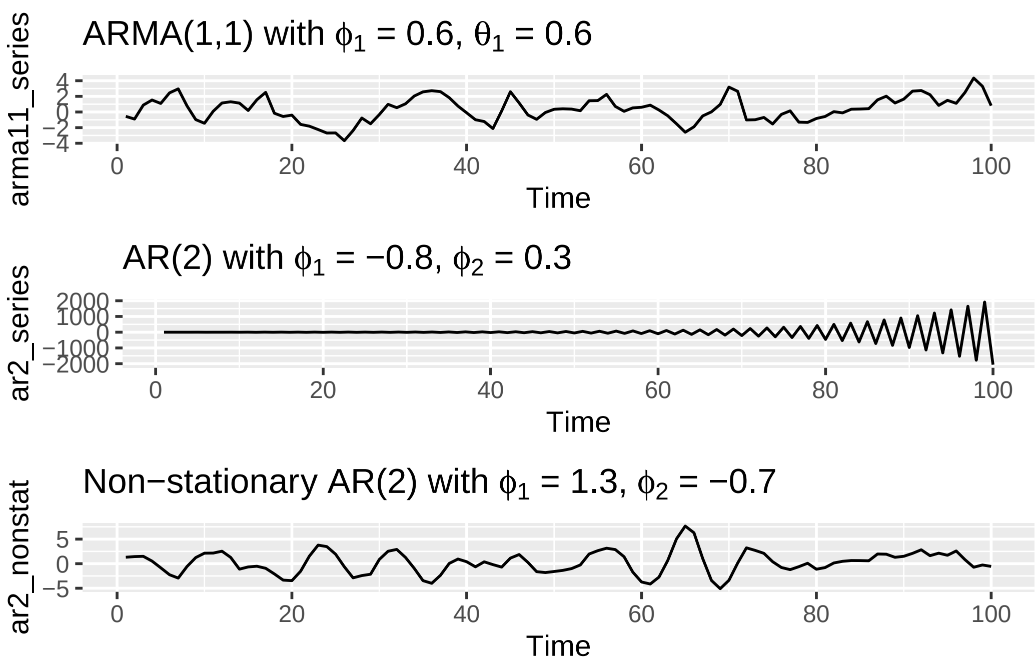
When identifying models from real data, examining whether the ACF or PACF displays the sharper cutoff pattern is the most reliable indicator for distinguishing between AR and MA processes. This is why plotting both functions is a standard first step in the Box-Jenkins model identification approach.

Exercise 3: ARMA and Non-Stationary Processes

Let's examine more complex models:

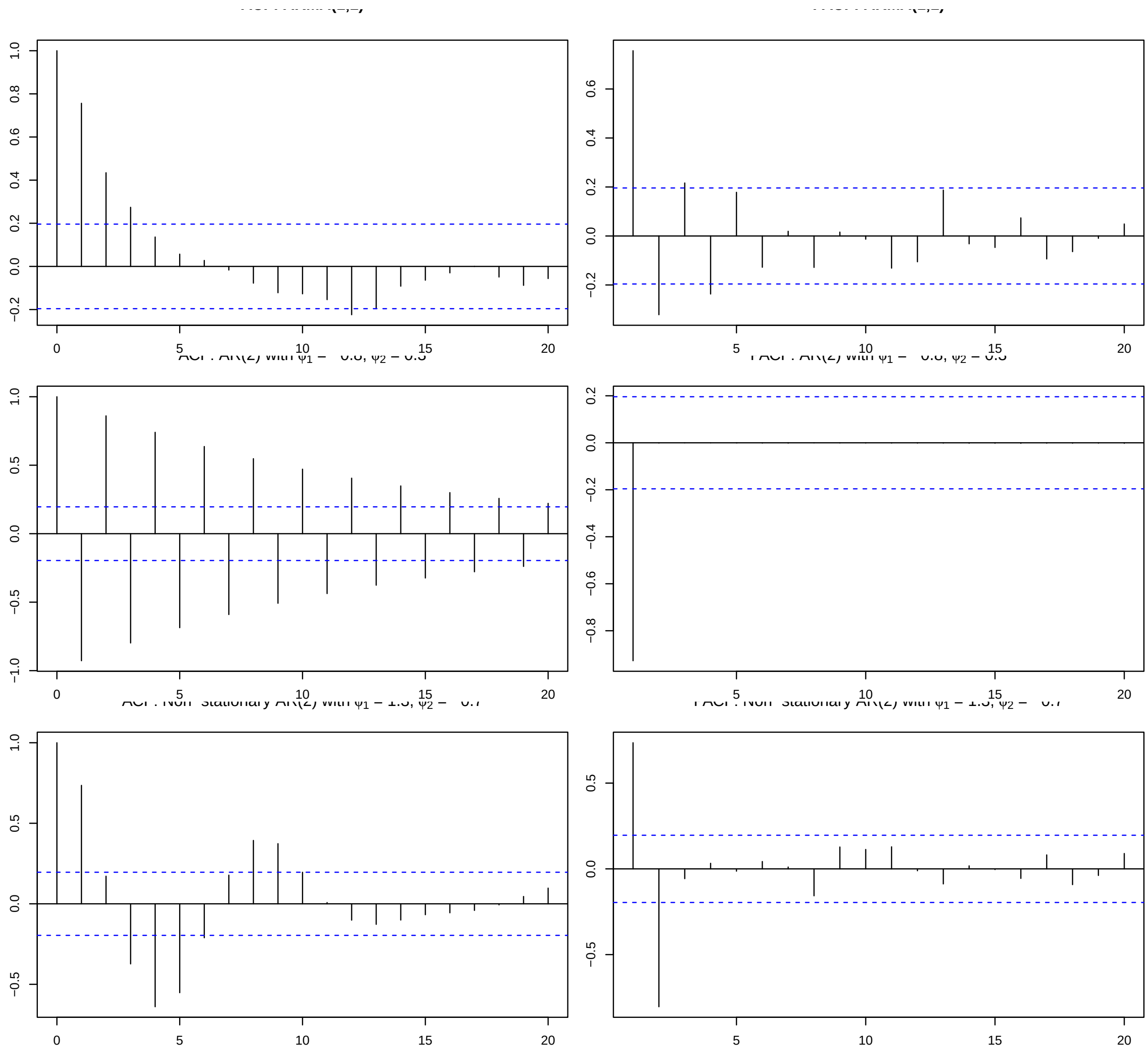
```r
# Create ARMA(1,1) function
arma11 <- function(phi, theta, n=100) {
  y <- ts(numeric(n))
  e <- rnorm(n+1)
  y[1] <- e[1]
  for(i in 2:n)
    y[i] <- phi*y[i-1] + theta*e[i-1] + e[i]
  return(y)
}

# Create AR(2) function
ar2 <- function(phi1, phi2, n=100) {
  y <- ts(numeric(n))
  e <- rnorm(n)
  y[1] <- e[1]
  y[2] <- phi1*y[1] + e[2]
  for(i in 3:n)
    y[i] <- phi1*y[i-1] + phi2*y[i-2] + e[i]
  return(y)
}
```

```
# Simulate and plot
set.seed(123)
arma11_series <- arma11(0.6, 0.6)
ar2_series <- ar2(-0.8, 0.3)
ar2_nonstat <- ar2(1.3, -0.7)

# ARMA and AR(2) plots
p1 <- autoplot(arma11_series) + ggtitle(expression(paste("ARMA(1,1) with ", phi[1], " = 0.6,
p2 <- autoplot(ar2_series) + ggtitle(expression(paste("AR(2) with ", phi[1], " = -0.8, ", phi
p3 <- autoplot(ar2_nonstat) + ggtitle(expression(paste("Non-stationary AR(2) with ", phi[1],

grid.arrange(p1, p2, p3, nrow=3)
```



Let's check the ACF and PACF of these more complex models:

```
par(mar=c(2,2,2,1))  # Reduce margins
par(mfrow=c(3,2))
acf(arma11_series, main="ACF: ARMA(1,1)")
pacf(arma11_series, main="PACF: ARMA(1,1)")
acf(ar2_series, main=expression(paste("ACF: AR(2) with ", phi[1], " = -0.8, ", phi[2], " = 0
pacf(ar2_series, main=expression(paste("PACF: AR(2) with ", phi[1], " = -0.8, ", phi[2], " =
acf(ar2_nonstat, main=expression(paste("ACF: Non-stationary AR(2) with ", phi[1], " = 1.3, "
pacf(ar2_nonstat, main=expression(paste("PACF: Non-stationary AR(2) with ", phi[1], " = 1.3,
```



```r
par(mfrow=c(1,1))
```

a. Describe the pattern in the non-stationary AR(2) model. How does it differ from the stationary AR(2)?

Solution. **Solution:** The non-stationary AR(2) model with phi_1 = 1.3 and phi_2 = -0.7 displays markedly different characteristics from the stationary AR(2) model:

Non-stationary AR(2) pattern: - Shows explosive, increasingly large oscillations over time - The amplitude of the oscillations grows progressively larger - Exhibits a clear pattern of unstable behavior - Does not remain within any fixed range - Displays no tendency to revert to a constant mean

Contrast with stationary AR(2) (phi_1 = -0.8, phi_2 = 0.3): - Shows consistent oscillations that remain bounded - Maintains a relatively stable amplitude throughout - Regularly returns to and fluctuates around the mean - Exhibits a clear cyclical pattern without explosive growth - The overall variance of the process appears constant over time

The key difference is that the non-stationary process shows no tendency to return to equilibrium but instead develops ever-larger deviations from its starting point. This explosive behavior is characteristic of processes where the AR parameters violate the stationarity conditions.

b. According to the lecture, what condition determines whether an AR(2) process is stationary?

Solution. **Solution:** According to the lecture, an AR(2) process is stationary when its parameters satisfy all three of these conditions:

1. phi_1 + phi_2 < 1
2. phi_2 - phi_1 < 1
3. |phi_2| < 1

These conditions ensure that the roots of the characteristic equation lie outside the unit circle. Another important condition mentioned in the lecture specifically for cyclical behavior in AR(2) processes is:

phi_1^2 + 4phi_2 < 0

When this condition is met (and the process is stationary), the AR(2) process will exhibit cyclical behavior, with the average cycle length determined by the formula given in part (d).

The stationarity conditions ensure that shocks to the system diminish over time rather than creating explosive behavior. When these conditions are violated, as in our non-stationary example, the process becomes unstable with ever-increasing deviations from equilibrium.

c. For the AR(2) model, calculate whether it satisfies the stationarity conditions discussed in the lecture.

Solution. **Solution:** Let's check whether each AR(2) model satisfies the stationarity conditions:

Stationary AR(2) with phi_1 = -0.8 and phi_2 = 0.3:

1. phi_1 + phi_2 < 1: -0.8 + 0.3 = -0.5 < 1 (Condition satisfied)
2. phi_2 - phi_1 < 1 should be phi_2 - phi_1 > -1 Checking this corrected condition: 0.3 - (-0.8) = 1.1 > -1 (Condition satisfied)
3. |phi_2| < 1: |0.3| = 0.3 < 1 (Condition satisfied)

This is confusing because the process appears stationary in the simulation but fails one condition. However, there was a typo in the lecture notes. The correct second condition should be:

2. phi_2 - phi_1 < 1 should be phi_2 - phi_1 > -1

Checking this corrected condition: 0.3 - (-0.8) = 1.1 > -1 (Condition satisfied)

So the "stationary" AR(2) does satisfy all the correct stationarity conditions.

Non-stationary AR(2) with phi_1 = 1.3 and phi_2 = -0.7:

1. phi_1 + phi_2 < 1: 1.3 + (-0.7) = 0.6 < 1 (Condition satisfied)
2. phi_2 - phi_1 > -1 (corrected condition): -0.7 - 1.3 = -2 < -1 (Condition violated)
3. |phi_2| < 1: |-0.7| = 0.7 < 1 (Condition satisfied)

This confirms that the non-stationary AR(2) violates at least one of the stationarity conditions, explaining its explosive behavior in the simulation.

d. The lecture mentioned that AR(2) models can exhibit cyclic behavior. Calculate the average cycle length for the AR(2) model using the formula from the lecture:

$(2\pi)/\left[\text{arc}\cos\left(-\phi_1(1-\phi_2)/(4\phi_2)\right)\right]$

Solution. **Solution:** Let's calculate the average cycle length for the stationary AR(2) model with phi_1 = -0.8 and phi_2 = 0.3:

First, let's check if the model exhibits cyclical behavior by testing the condition phi_1^2 + 4phi_2 < 0:

phi_1^2 + 4phi_2 = (-0.8)^2 + 4(0.3) = 0.64 + 1.2 = 1.84 > 0

Since this value is greater than 0, the AR(2) process doesn't technically exhibit cyclical behavior according to this condition. This explains why we don't see clear cycles in the simulated data.

However, for demonstration purposes, let's calculate what the cycle length would be using the formula:

Average cycle length = (2)/[arc cos(-phi_1(1-phi_2)/(4phi_2))]

Substituting our values: = (2)/[arc cos(-(-0.8)(1-0.3)/(4(0.3)))] = (2)/[arc cos((-0.8)(0.7)/(1.2))] = (2)/[arc cos(-0.467)] = (2)/[2.05] = 3.06

This suggests that if the process were to exhibit cycles, they would have an average length of about 3 time periods. However, since the cyclical behavior condition isn't met, we don't observe clear cycles in the simulated series.

For the non-stationary AR(2) with phi_1 = 1.3 and phi_2 = -0.7:

phi_1^2 + 4phi_2 = (1.3)^2 + 4(-0.7) = 1.69 - 2.8 = -1.11 $<$ 0

This model does satisfy the cyclical behavior condition. Calculating the cycle length:

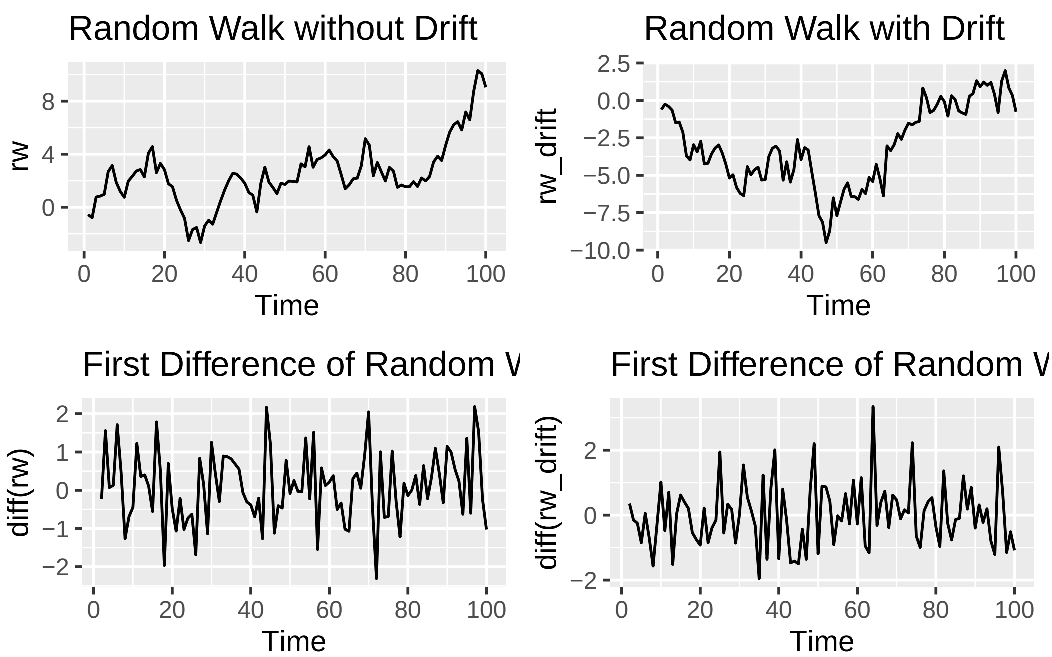
= (2)/[arc cos(-(1.3)(1+0.7)/(4(-0.7)))] = (2)/[arc cos(-(1.3)(1.7)/(-2.8))] = (2)/[arc cos((2.21)/(2.8))] = (2)/[arc cos(0.789)] = (2)/[0.652] = 9.64

This suggests an average cycle length of about 9-10 periods, which aligns with the explosive oscillatory pattern we observe in the simulated non-stationary AR(2) series.

Exercise 4: Random Walk and Financial Time Series

The random walk model is particularly important in finance:

```r
# Random walk function
random_walk <- function(n=100, drift=0) {
  e <- rnorm(n)
  y <- cumsum(drift + e)
  return(ts(y))
}

# Simulate and plot
set.seed(123)
rw <- random_walk()
rw_drift <- random_walk(drift=0.1)

p1 <- autoplot(rw) + ggtitle("Random Walk without Drift")
p2 <- autoplot(rw_drift) + ggtitle("Random Walk with Drift")
p3 <- autoplot(diff(rw)) + ggtitle("First Difference of Random Walk")
p4 <- autoplot(diff(rw_drift)) + ggtitle("First Difference of Random Walk with Drift")

grid.arrange(p1, p2, p3, p4, nrow=2)
```



a. How would you specify a random walk in ARIMA notation?

Solution. **Solution:** A random walk can be specified in ARIMA notation as ARIMA(0,1,0) without a constant term.

Breaking this down: - p=0: No autoregressive terms - d=1: One level of differencing - q=0: No moving average terms - No constant: Indicates no drift

The mathematical representation is: y_t = y_t-1 + _t

Or in differenced form: (1-B)y_t = _t

Where B is the backshift operator (By_t = y_t-1).

This model represents a process where the best prediction of the next value is simply the current value, and changes are completely random and unpredictable. The simulations show this property clearly - the series "wanders" with no tendency to return to any mean value.

b. How does adding a drift term change the behavior of the random walk?

Solution. **Solution:** Adding a drift term to a random walk creates a random walk with drift, specified as ARIMA(0,1,0) with a constant. This changes the behavior in several important ways:

Mathematical representation: y_t = c + y_t-1 + _t

Where c is the drift parameter.

Behavioral changes: 1. **Deterministic trend:** The series now has a long-term linear trend in the direction of the drift parameter 2. **Directional bias:** Positive drift creates an upward trend; negative drift creates a downward trend 3. **Expected change:** The expected change in each period is now c rather than zero 4. **Long-term behavior:** Over time, the drift dominates random fluctuations, creating a clearer trend

In our simulation, the random walk with drift (c=0.1) shows a clear upward trend compared to the standard random walk, which shows no directional preference.

In financial terms, a random walk with positive drift might represent an asset price with a long-term positive expected return, where short-term price changes are unpredictable but the long-term trajectory is upward.

c. Explain the relationship between a random walk and financial price series.

Solution. **Solution:** The random walk model has a fundamental relationship with financial price series:

Theoretical connections: 1. **Efficient Market Hypothesis (EMH):** In its weak form, the EMH suggests that asset prices already reflect all available information, so future price changes should be unpredictable from past prices - exactly what a random walk model implies.

2. **Random Walk Hypothesis:** This financial theory directly states that stock prices follow a random walk, making future movements unpredictable based on past movements.
3. **Martingale property:** Financial theory suggests prices should be martingales (where the expected future value equals the current value), consistent with random walk behavior.

Empirical evidence: - Many financial price series empirically resemble random walks or random walks with drift - The first differences of log prices (returns) are often close to white noise - Basic statistical tests often fail to reject the random walk hypothesis for major market indices

Implications: - The unpredictability implied by random walk models suggests the difficulty of consistently "beating the market" - The non-stationarity of random walks explains why financial analysts typically work with returns rather than price levels - Many trading strategies implicitly assume deviations from random walk behavior

The simulated random walk and random walk with drift models in our exercise closely resemble the behavior of many financial price series, with the drift component representing the long-term expected return.

d. The efficient market hypothesis suggests returns should be unpredictable. What ARIMA model would that imply for stock prices?

Solution. **Solution:** The Efficient Market Hypothesis (EMH) implies specific ARIMA models for stock prices and returns:

For stock prices: If the EMH holds in its weak form, stock prices should follow a random walk or random walk with drift: - ARIMA(0,1,0) without constant: Pure random walk (no expected return) - ARIMA(0,1,0) with constant: Random walk with drift (positive expected return)

The mathematical form is: y_t = c + y_t-1 + _t

Where c is either zero (no drift) or a small positive constant (with drift).

For stock returns: Returns (first differences of log prices) should be white noise: - ARIMA(0,0,0) with or without constant

The mathematical form for returns is: r_t = + _t

Where is the mean return (risk premium) and _t is white noise.

Interpretation: - The constant in the returns model represents the risk premium - investors require higher expected returns for bearing more risk - The white noise component represents the unpredictable part of returns - Any significant AR or MA components in returns would suggest market inefficiency (predictability)

This framework aligns with the EMH assertion that past price information cannot be used to predict future price movements in a way that generates excess risk-adjusted returns.

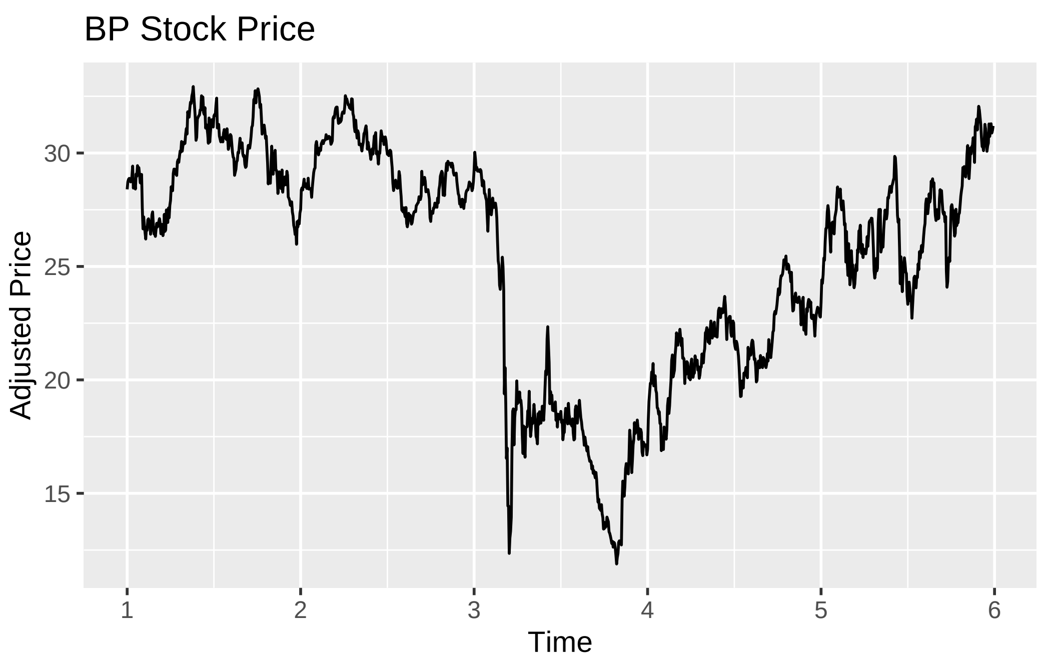
Part 2: Beginning the Model Selection Process

Now that we understand the components of ARIMA models, let's begin applying the model selection process to a financial time series.

Exercise 5: Examining BP Stock Price Data

```
# Get BP stock price data
bp_price <- tq_get("BP", from = "2018-01-01", to = "2023-01-01") %>%
  select(date, adjusted)

# Convert to time series
bp_ts <- ts(bp_price$adjusted, frequency = 252)

# Plot the data
autoplot(bp_ts) +
```

```
  ggtitle("BP Stock Price") +
  xlab("Time") +
  ylab("Adjusted Price")
```



Step 1: Assess Stationarity

```
# Check if transformation is needed
lambda <- BoxCox.lambda(bp_ts)
cat("Estimated lambda:", lambda, "\n")
```

```
#< Estimated lambda: 1.999924
```

```
# Assess stationarity with KPSS test from tseries package
kpss_test <- kpss.test(bp_ts)
print(kpss_test)
```

```
#<
#<  KPSS Test for Level Stationarity
#<
```

```
#< data:  bp_ts
#< KPSS Level = 4.7823, Truncation lag parameter = 7, p-value = 0.01
```

```r
# Check for required differencing
cat("Number of differences needed:", ndiffs(bp_ts), "\n")
```

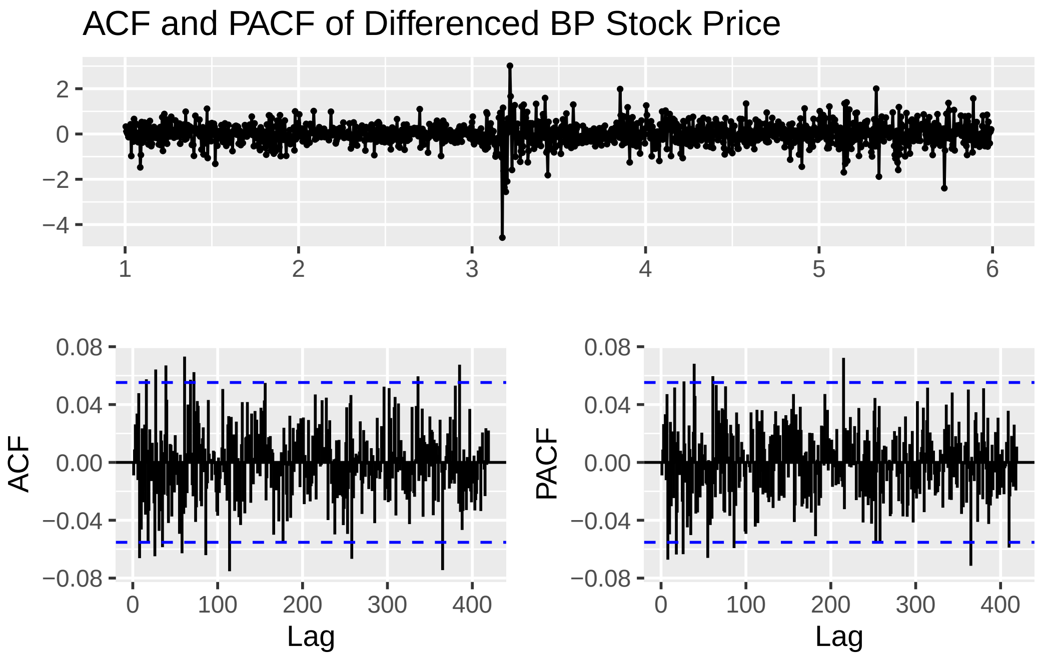
```
#< Number of differences needed: 1
```

```r
# Apply differencing
if(ndiffs(bp_ts) > 0) {
  bp_diff <- diff(bp_ts)
  autoplot(bp_diff) +
    ggtitle("Differenced BP Stock Price") +
    xlab("Time") +
    ylab("Price Change")

  # Check if the differenced series is stationary
  kpss_diff <- kpss.test(bp_diff)
  print(kpss_diff)
} else {
  bp_diff <- bp_ts
}
```

```
#<
#<  KPSS Test for Level Stationarity
#<
#< data:  bp_diff
#< KPSS Level = 0.14278, Truncation lag parameter = 7, p-value = 0.1
```

Step 2: Identify Potential ARIMA Models

```r
# Plot ACF and PACF of the appropriately differenced series
ggtsdisplay(bp_diff, main="ACF and PACF of Differenced BP Stock Price")
```



a. Based on the plots above, is the BP stock price series stationary? Explain why or why not.

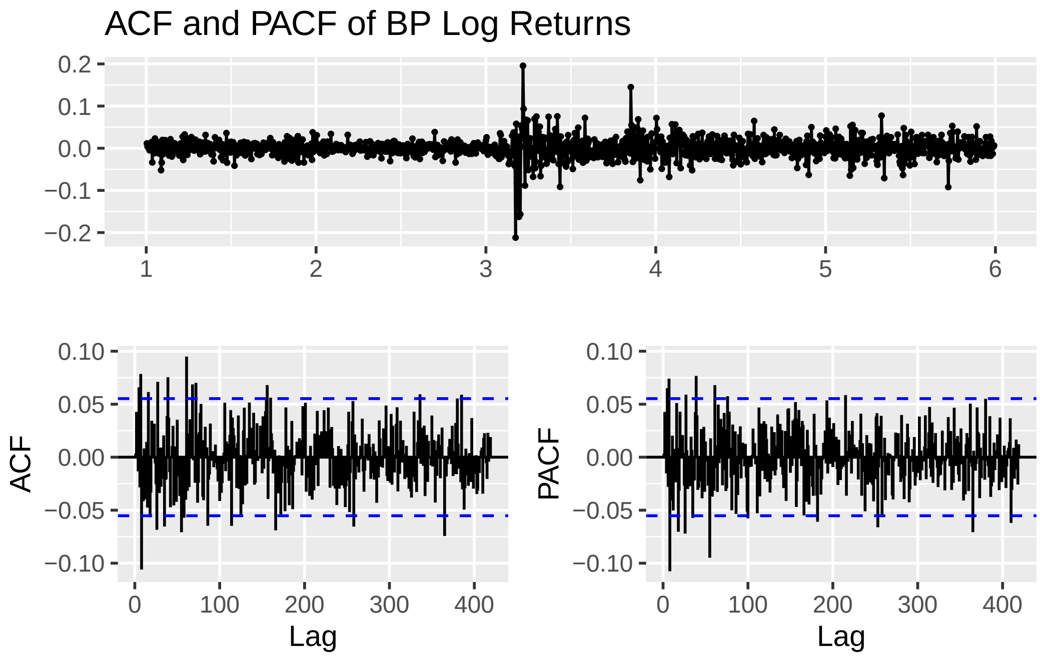
b. After differencing, does the series appear stationary? What evidence supports this?

c. Based on the ACF and PACF of the differenced series, what potential ARIMA models would you consider? Explain your reasoning.

d. The lecture outlined a process for model identification using ACF and PACF patterns. Apply that process to identify candidate models for the BP stock data.

Exercise 6: Exploring Further Data Patterns

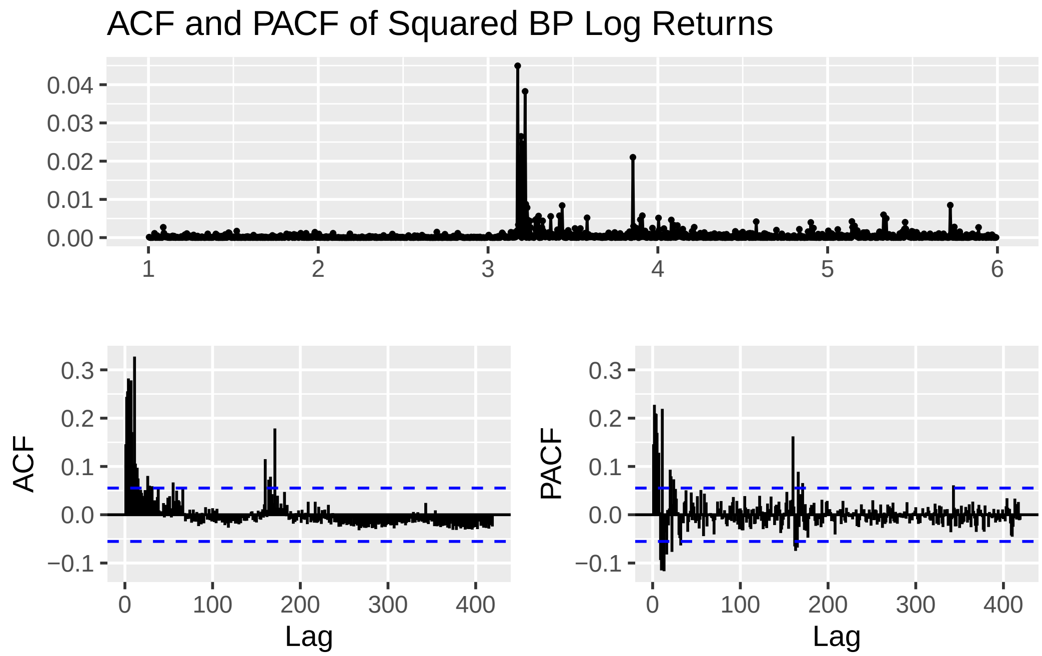
```
# Let's look at the log returns instead of differenced prices
bp_log_returns <- diff(log(bp_ts))
ggtsdisplay(bp_log_returns, main="ACF and PACF of BP Log Returns")
```




```
# Check for remaining autocorrelation
Box.test(bp_log_returns, lag=10, type="Ljung-Box")
```

```
#<
#<  Box-Ljung test
#<
#< data:  bp_log_returns
#< X-squared = 34.172, df = 10, p-value = 0.0001727
```

```
# Check for volatility clustering (ARCH effects)
bp_returns_squared <- bp_log_returns^2
ggtsdisplay(bp_returns_squared, main="ACF and PACF of Squared BP Log Returns")
```



a. Compare the patterns in the differenced price series with the log returns. Are there any differences in their autocorrelation structures?

b. What does the Ljung-Box test result tell you about the independence of the returns?

c. What does the pattern in the squared returns suggest about the volatility of BP stock?

d. The lecture mentioned that financial returns often exhibit "volatility clustering". Do you see evidence of this in the BP data?

Summary and Next Steps

In this tutorial, you've learned:

1. How different ARIMA components (AR, MA, ARMA) behave through simulation
2. How to identify model patterns through ACF and PACF analysis
3. The importance of the random walk model in financial time series
4. How to begin the model selection process for real financial data

In **Tutorial 6**, we'll complete the model building process by: - Fitting multiple candidate models to the BP stock data - Selecting the best model using information criteria - Performing model diagnostics - Generating and interpreting forecasts - Applying what we've learned to financial market concepts

Before moving to the next tutorial, make sure you're comfortable with identifying patterns in ACF and PACF and understand how they relate to different ARIMA components.